Estimating Elastic Buckling Load for an Axially Loaded Column Bolted to a Simply Supported Plate using Energy Method

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Abstract

This paper deals with the elastic stability of a column bolted at its mid-height to a simply supported square plate and subjected to a concentrated load, using energy method. A uniform homogeneous column is assumed to be pinned at both ends. From symmetry considerations, half of the column is modeled by making the plate acting as a torsion spring on the column at its mid-height. The column length and cross-section, plate dimensions and thickness, and the material properties for the column and the plate catch the interest of the author. The problem is solved by using energy method and ultimately, the elastic buckling load is found. The analytical elastic buckling load is compared with a numerical solution obtained from finite element method using SAP2000. The numerical results agree with the analytical solution. The finite element model is refined to catch the actual effect of the bolted plate on the elastic buckling load. It has been found that the elastic buckling load is increased due to the increase in the rotational stiffness provided from the plate.

Keywords: Column, Elastic Buckling, Axial Load, Energy Method, FEM.

1. Introduction

Buckling is an important consideration in structural design. In some cases, it governs the design before the strength criterion does especially when the member is slender and lightweight [1]. There are two general approaches in finding the elastic buckling load: a) the vector approach and (b) the energy approach [2,3].

Solutions of simple cases of buckling are given by Timoshenko and Gere [4]. Wang et al. [1] use the vector approach to give exact solutions for buckling of various structural members. In contrary, some exact solutions for columns with variable cross-section are provided in terms of Lomel functions by Elishakoff and Pelligrini [5,6].

Atay and Coskun [7] analyze Euler columns with a continuous elastic restraint using variational iteration method (VIM). However, Basbuk et al. [8] use the homotopy analysis method (HAM) to find the critical buckling load for Euler columns with elastic ends restraints. Also, the HAM method was used by Eryilmaz [9] to find the buckling load of Euler columns with a continuous elastic restraint.

Sampaio et al. [10] gives the solution for buckling behavior of inclined beam-column, using energy method. Similarly, the energy approach was used by Zdravkovic et al. [11] to study the buckling load of a three-segment stepped column subjected to an axial load. Rychlewska [12] provide numerical solutions for axially functionally graded Euler-Bernoulli beam for various boundary conditions. In this paper, the energy approach is used to find the elastic buckling load of a column with both ends pinned and bolted to a simply supported plate at its mid-height. The exact solution from this work is compared with the analytical solution provided by Wang et al. In addition, a finite element solution is performed in this paper to compare the results and to study the effect of the bolted plate on the buckling load.

2. Problem Definition

The case studied in this paper is shown in Figure (1-a). The column is bolted to a simply supported plate at its mid-height. The column length is ($2L$) and the plate is assumed square with a dimension (a). For the purpose of simplicity in calculations shown later, the plate dimension is assumed equal to ($1.2L$). The column and the plate materials are assumed to be the same. The modulus of elasticity of the material is (E). The column cross-section is shown in Figure (1-b). For the purpose of mathematical simplicity, the plate thickness is assumed to be the same as the column width (b).

The problem is simplified by taking advantage of symmetry and considering the plate effective in the x-direction only and acting as a torsion spring at the middle of the column. Therefore, the stiffness of the torsion spring is determined by shrinking the y-direction of the plate to a width equal to the column depth (h) and treats the plate as a simply supported beam in the x-direction. Based on the above assumption and since the thickness of the plate is taken as the column width (b), the plate and the column is now having the same moment of inertia (I). Note that buckling is governed by the weak-axis buckling; therefore, the moment of inertia is ($hb^3/12$). To get the rotational stiffness, the simply supported beam is subjected to an arbitrary torque (T) at its mid-span. Hence, the rotational stiffness (k_T) at the middle of the beam represents the stiffness of the torsion spring of the column (k_T=12EI/a). The actual stiffness of the torsion spring is greater than the one obtained above because of the two dimensional behavior of the plate. Therefore, using this approximation is conservative. The effect of this approximation is studied in the numerical results section.

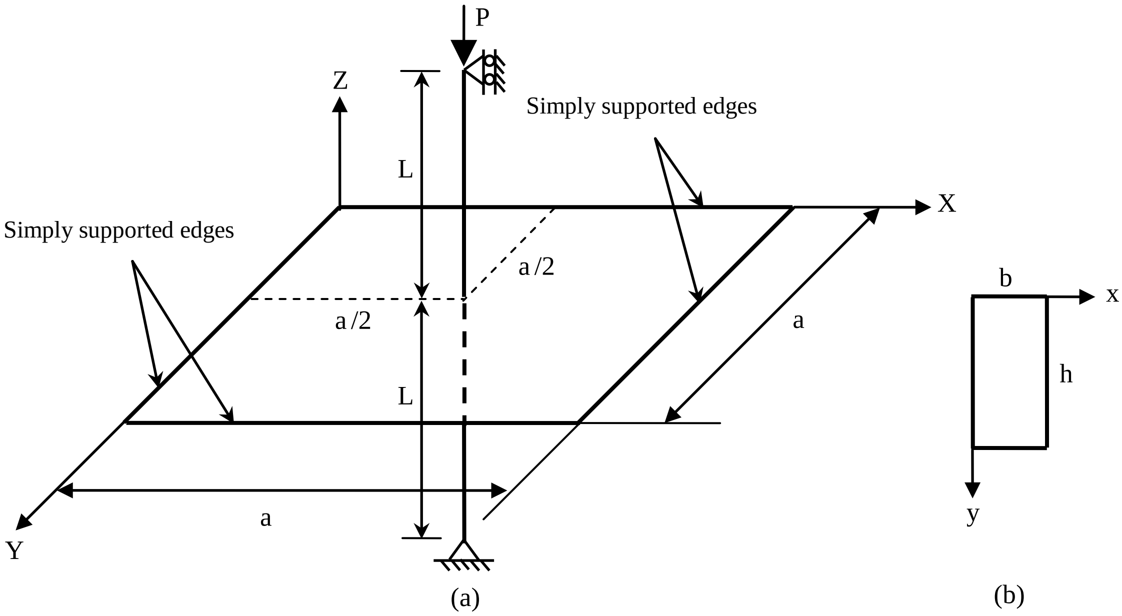

Figure 1: (a) Pinned ends column bolted to a simply supported plate (b) Column Cross-section

The simplified problem is shown in Figure (2). The formulation of the problem analytically using the energy approach is explained in the upcoming section.

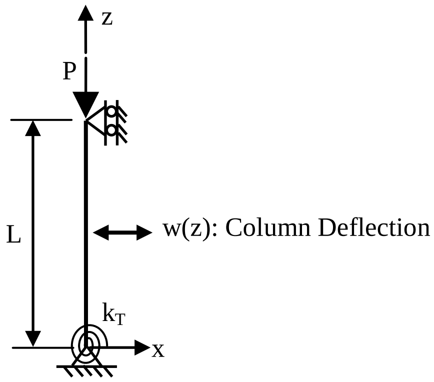

Figure 2: Pinned ends column with a torsional spring at the lower support (upper column)

3. Energy Method Formulation

The model in Figure (2) is analyzed using the energy method approach. This method minimizes the total energy (which is the sum of internal energy and potential energy due to the loads) to obtain the governing equations [1,2]. The internal energy of the model is divided into two parts, the strain energy due to the flexure effects (U_1) and the potential energy stored in the torsion spring (U_2). The strain energy due to the flexure effects is given by [1]:

$$U_1 = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx \tag{1}$$

The potential energy stored in the spring is given by:

$$U_2 = \frac{1}{2}k_T\left(\theta\big|_{z=0}\right)^2 = \frac{1}{2}k_T\left(\frac{dw}{dx}\bigg|_{z=0}\right)^2 \tag{2}$$

The work done by the external load (P) is given by [1]:

$$W = \frac{1}{2}\int_0^L P\left(\frac{dw}{dx}\right)^2 dx \tag{3}$$

By summing up Equations (1), (2), and (3) and noting that the potential energy (V) has a reverse sign of the work (W), the total energy of the system shown in Figure (2) is:

$$\pi(w) = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx - \frac{1}{2}\int_0^L P\left(\frac{dw}{dx}\right)^2 dx + \frac{1}{2}k_T\left(\frac{dw}{dx}\bigg|_{z=0}\right)^2 \tag{4}$$

To get the differential equation (Euler-Lagrange Equation) and the boundary conditions, the variation of the total energy of the system is minimized as follows:

$$\delta\pi(w)=\int_0^L EI\,\frac{d^2w}{dx^2}\frac{d^2}{dx^2}(\delta w)\;dx$$
$$-\int_0^L P\;\frac{dw}{dx}\frac{d}{dx}(\delta w)\;dx+k_T\,\frac{dw}{dx}\delta\left(\left.\frac{dw}{dx}\right|_{z=0}\right)\qquad(5)$$

Integrating by parts for the first two terms of Equation (5), yields:

$$\int_0^L\left[\frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2}\right)+P\;\frac{d^2w}{dx^2}\right]\delta w\;dx$$
$$-\left[P\frac{dw}{dx}+\frac{d}{dx}\left(EI\,\frac{d^2w}{dx^2}\right)\right]\delta w\Bigg|_0^L$$
$$+\left[-EI\,\frac{d^2w}{dx^2}+k_T\,\frac{dw}{dx}\right]\delta\left(\frac{dw}{dx}\right)\Bigg|_{z=0}$$
$$+\left[EI\,\frac{d^2w}{dx^2}\delta\left(\frac{dw}{dx}\right)\right]\Bigg|_{z=L}=0\qquad(6)$$

Making use of Equation (6) and applying the fundamental Lemma [1], the differential equation of the system is given by:

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2}\right)+P\;\frac{d^2w}{dx^2}=0\qquad(7)$$

Similarly, the boundary conditions are as follows:

$$\left[P\frac{dw}{dx}+\frac{d}{dx}\left(EI\,\frac{d^2w}{dx^2}\right)\right]\delta w\Bigg|_0^L=0\qquad(8\text{-a})$$

$$\left[-EI\,\frac{d^2w}{dx^2}+k_T\,\frac{dw}{dx}\right]\delta\left(\frac{dw}{dx}\right)\Bigg|_{z=0}=0\qquad(8\text{-b})$$

$$EI\,\frac{d^2w}{dx^2}\delta\left(\frac{dw}{dx}\right)\Bigg|_{z=L}=0\qquad(8\text{-c})$$

For the model studied here, the following boundary conditions are applied:

$$w(0)=0,\quad w(L)=0\qquad(9\text{-a})$$

$$EI\,\frac{d^2w}{dx^2}\bigg|_{z=0}=k_T\,\frac{dw}{dx}\bigg|_{z=0}=T,$$
$$EI\,\frac{d^2w}{dx^2}\bigg|_{z=L}=0\qquad(9\text{-b})$$

It is worth mentioning that the boundary conditions in Equation (9-a) are called essential boundary conditions; however, the boundary conditions in Equation (9-b) are called natural boundary conditions.

4. Analytical Results

The differential equation of the system and the boundary conditions are obtained in the previous section. In this section, the estimated elastic buckling load (P_{cr}) for the model shown in Figure (1) is obtained by using an exact approach. For a constant cross-section, Equation (7) has a general solution of the form given by [1]:

$$w(z)=A+Bx+C\cos\;(\lambda x)+D\sin(\lambda x)\quad(10\text{-a})$$

where

$$\lambda^2=\frac{P}{EI}\qquad(10\text{-b})$$

Applying the boundary conditions from Equation (9) into Equation (10-a) leads to:

$$B\;\left[\sin(\lambda L)-\frac{\beta\lambda L}{\lambda^2L^2+\beta}\cos\;(\lambda L)\right]=0\qquad(11)$$

in which the factor (β) is obtained by setting the rotational stiffness (k_T) in the from:

$$k_T=\frac{12EI}{a}=\frac{\beta EI}{L}\qquad(12)$$

Assuming ($\alpha=\lambda L$), where α is a critical load factor. For a non-trivial solution, the following transcendental function is obtained:

$$\tan(\alpha)=\frac{\beta\alpha}{\alpha^2+\beta}\qquad(13)$$

The critical buckling load is related to the first root of the critical load factor α. To get the factor α, Equation (13) is plotted using MATLAB [13]. For (β=10), the first root represents the first point of intersection between the two functions tan(α) and ($\beta\ \alpha\ /\ \alpha^2+\beta$) as shown in Figure (3).

The value of the critical load factor α is obtained from Figure (3) as (α_{cr}=4.1323). Substituting α_{cr} as ($\lambda_{cr}\ L$) and making use of Equation (10-b), the first buckling load is given by:

$$P_{cr}=\frac{{\alpha_{cr}}^2EI}{L^2}=\frac{17.076EI}{L^2}\qquad(14)$$

As mentioned previously, Wang et al. [4] used the vector approach to get the critical buckling loads of such cases. The buckling load from Equation (14) is compared with results provided by Wang et al. [4]. The exact same value of the critical load factor is obtained here in this work using the energy method.

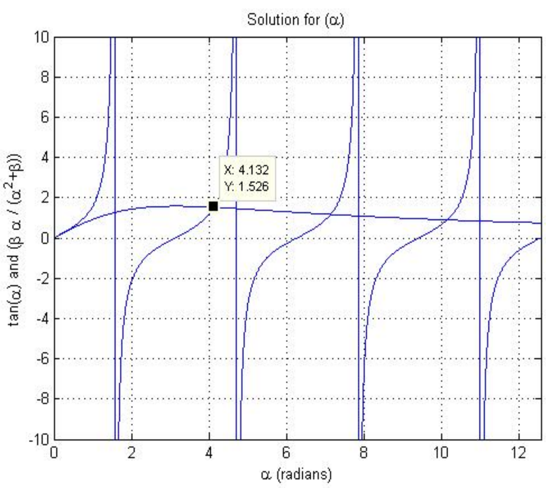

Figure 3: Critical load factor α_{cr}

5. Numerical Results

To compare the results and study the effect of the bolted plate on the elastic buckling load, a numerical solution is obtained from finite element method. To do this, the finite element program SAP2000 [14] is used. A 4.8mx4.8m square normal concrete slab simply supported at all edges is considered. As discussed above, the slab thickness is assumed to be the same as the least dimension of the column. The column is a normal concrete with dimensions of 400mm x 200mm. The column is 4m length (L=a/1.2=4.8/1.2=4m). The concrete compressive strength for both the column and the slab is assumed as (f'_c=25 MPa). According to the ACI Code [15], the modulus of elasticity equation for a normal weight concrete is ($E_c=4700\sqrt{f'_c}$). For the specified compressive strength of concrete, the modulus of elasticity is calculated as 23500 MPa. A 3-D model is considered in SAP2000 to model the buckling behavior for the case of interest shown in Figure -1-. A 4-node quadrilateral shell element with six degrees of freedom at each node is used to model the slab with a total number of 256 elements.

The self-weight of the column and the slab is ignored in the SAP2000 model to be consistent with analytical model shown in Figure (2). The shear deformation is ignored and only flexural deformation is considered. In addition, the simply supported plate is considered to be effective in the direction of buckling only (i.e. ignoring the two-way effects of the plate). Buckling analysis is performed and the buckling load is obtained. A snap shot is taken from the SAP2000 window to show the SAP2000 model and the critical buckling load value as shown in Figure (4).

The estimated critical buckling load from Equation (14) using the dimensions and material properties given above is 6688 kN. The finite element analysis based on the same assumptions used in the analytical model gives a critical buckling load of 6804 kN. The difference between the analytical results and the numerical results is only 2%. Hence, the numerical results are in excellent agreement with analytical results.

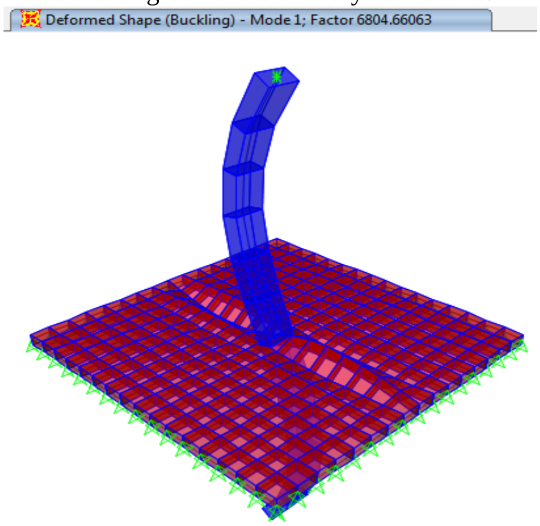

Figure 4: A 3-D finite element model in SAP2000: first buckling analysis

It can be noted that the actual rotational stiffness provided from the simply supported plate is much higher than the one assumed in the analytical and the numerical models. In other words, a higher critical buckling load can be obtained. To perfectly model the actual rotational stiffness, the two way flexural effects of the slab and the torsional stiffness due to the twisting moment in the slab are taken into consideration. For this purpose, a second buckling analysis is performed and the value of the critical buckling load is 10335 kN as shown in Figure (5). The critical buckling load is now higher than the value from the first buckling analysis by 51%.

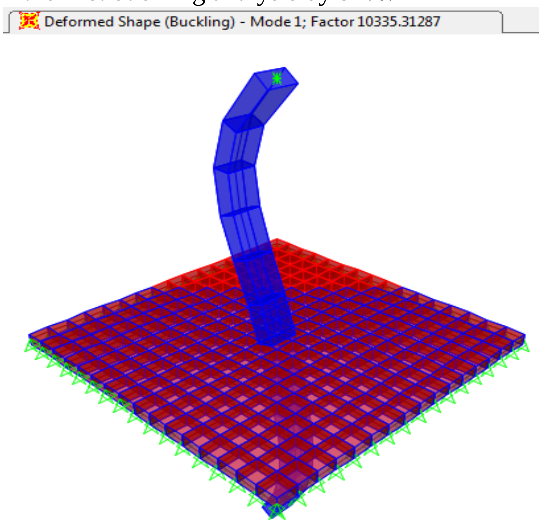

Figure 5: A 3-D finite element model in SAP2000: second buckling analysis

6. Conclusions

It has been found that the analytical solution using the energy approach is identical with the analytical solution provided by Wang et al. [4] using the vector approach. Also, the numerical solution from SAP2000 shows an excellent agreement with the analytical solution.

To get a better solution, the simply supported plate is modeled by considering flexure in both directions. In addition, the torsional stiffness from the twisting moment in the slab is taken into consideration. For this purpose, a second elastic buckling analysis is performed by using SAP2000. The critical buckling load is increased by (51%) as compared with the first buckling analysis. This happens because the rotational stiffness provided from the slab is increased. Hence, the torsion stiffness from the slab has a significant effect on the column buckling.

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تقدير حمل الانبعاج المرن لعمود محمل محوريا ومربوط ببلاطة بسيطة الاسناد باستخدام طريقة الطاقة

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الخلاصة

تتناول هذه الدراسة الاستقرار المرن لعمود مربوط بلوح مربع بسيط الاسناد في منتصف الارتفاع ومعرض الى حمل مركز، وذلك باستخدام طريقة الطاقة، حيث تم فرض ان العمود متجانس ومسمر في نهايتيه. من خلال اخذ التماثل بنظر الاعتبار، تم تمثيل نصف العمود فقط وجعل البلاطة عبارة عن نابض (لي) يؤثر على العمود في منتصف الارتفاع. ان طول العمود وابعاد المقطع وابعاد البلاطة وسمكها ومواصفات مادة العمود والبلاطة هي حالة جلبت انتباه الباحث. تم تحليل المسالة باستخدام طريقة الطاقة وفي النهاية تم الحصول على حمل الانبعاج المرن. تمت ايضا مقارنة الحمل التحليلي المرن مع الحل العددي الذي تم الحصول عليه بواسطة طريقة العناصر المحددة باستخدام برنامج SAP2000. اظهرت النتائج العددية توافق تام مع الحل التحليلي. اخيرا تم صقل المسالة لدراسة التاثير الحقيقي للبلاطة على حمل الانبعاج المرن حيث وجد ان حمل الانبعاج المرن قد ازداد نتيجة لزيادة الصلابة الدورانية في البلاطة.